Supplementary Appendix to
Interest Rate Dynamics and Commodity Prices

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Appendix D. An Identification Equivalence Result

Consider an economy E with linear inverse demand function $p(x) = a + dx$ where $a > 0$ and $d < 0$. Let $\{Y_t\}$ be a stationary Markov process with transition probability Ψ. Let b be the lower bound of the total available supply in this economy.

Let $\tilde{E}$ be another economy where the output process satisfies $\tilde{Y}_t = \mu + \sigma Y_t$ with $\sigma > 0$ and the transition probability of $\{\tilde{Y}_t\}$ satisfies[1]

$$\tilde{\Psi}(y, \hat{Y}) = \Psi\left(\frac{y-\mu}{\sigma}, \frac{\hat{Y}-\mu}{\sigma}\right). \tag{D.1}$$

Moreover, let the lower bound of the total available supply of economy $\tilde{E}$ be $\tilde{b} = \mu + \sigma b$ and the inverse demand function be

$$\tilde{p}(x) = \left(a - \frac{d\mu}{\sigma}\right) + \frac{d}{\sigma}x. \tag{D.2}$$

The remaining assumptions are the same across economies E and $\tilde{E}$.

Proposition D.1. *$\tilde{E}$ and E generate the same commodity price process.*

Proof. To simplify notation, let f and i be the equilibrium pricing function and the equilibrium inventory function of the baseline economy E. Without loss of generality, we may assume $Z_t = Y_t$.[2] Then for all $(x, y) \in \mathsf{S}$, $\{f(x,y), i(x,y)\}$ is the unique solution to

$$f(x,y) = \min\left\{\max\left\{e^{-\delta}\,\mathbb{E}_y\,\hat{M} f[e^{-\delta} i(x,y) + \hat{Y}, \hat{Y}] - k,\ p(x)\right\}, p(b)\right\} \tag{D.3}$$

$$i(x,y) = \begin{cases} x - p^{-1}[f(x,y)], & x < x_f(y) \\ x_f(y) - p^{-1}(0), & x \geq x_f(y) \end{cases} \tag{D.4}$$

where

$$x_f(y) := \inf\left\{x \geq p^{-1}(0) : f(x,y) = 0\right\}.$$

[1] Condition (D.1) obviously holds if, for example, $\{Y_t\}$ is IID and follows a truncated normal distribution with mean μ_0, variance σ_0^2, and truncation thresholds $y_l < y_u$. Because in this case, $\{\tilde{Y}_t\}$ is IID and follows a truncated normal distribution as well, with mean $\mu + \sigma\mu_0$, variance $\sigma^2\sigma_0^2$, and truncation thresholds $\mu + \sigma y_l < \mu + \sigma y_u$. Note that (D.1) does not hold if $\{Y_t\}$ and $\{\tilde{Y}_t\}$ do not follow the same *type* of distribution. For example, it does not hold if $\{Y_t\}$ is IID lognormally distributed, since $\{\tilde{Y}_t\}$ is not lognormally distributed as a linear transform of $\{Y_t\}$.

[2] In general, Z_t is a multivariate Markov process and Y_t corresponds to one dimension of Z_t.

Consider economy $\tilde{E}$, where all magnitudes are denoted with tildes. Let

$$\tilde{x} = \mu + \sigma x, \quad \tilde{y} = \mu + \sigma y,$$
$$\tilde{m}(\tilde{y}, \varepsilon) = m(y, \varepsilon), \quad \tilde{f}(\tilde{x}, \tilde{y}) = f(x, y), \quad \tilde{\imath}(\tilde{x}, \tilde{y}) = \sigma\, i(x, y). \tag{D.5}$$

To prove the statement of the proposition, it suffices to show that $\{\tilde{f}(\tilde{x}, \tilde{y}), \tilde{\imath}(\tilde{x}, \tilde{y})\}$ is the unique solution to

$$\tilde{f}(\tilde{x}, \tilde{y}) = \min\left\{\max\left\{\mathrm{e}^{-\delta}\, \mathbb{E}_{\tilde{y}}\, \hat{\tilde{M}} \tilde{f}[\mathrm{e}^{-\delta}\tilde{\imath}(\tilde{x}, \tilde{y}) + \hat{\tilde{Y}}, \hat{\tilde{Y}}], \tilde{p}(\tilde{x})\right\} - k, \tilde{p}(\tilde{b})\right\} \tag{D.6}$$

$$\tilde{\imath}(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{x} - \tilde{p}^{-1}[\tilde{f}(\tilde{x}, \tilde{y})], & \tilde{x} > x_{\tilde{f}}(\tilde{y}) \\ x_{\tilde{f}}(\tilde{y}) - \tilde{p}^{-1}(0), & \tilde{x} \leq x_{\tilde{f}}(\tilde{y}) \end{cases} \tag{D.7}$$

where

$$x_{\tilde{f}}(\tilde{y}) := \inf\left\{\tilde{x} \geq \tilde{p}^{-1}(0) : \tilde{f}(\tilde{x}, \tilde{y}) = 0\right\}.$$

This is true by referring to (D.3)–(D.4). In particular, by (D.2) and (D.5),

$$\hat{\tilde{M}} = \tilde{m}(\hat{\tilde{Y}}, \hat{\varepsilon}) = m(\hat{Y}, \hat{\varepsilon}) = \hat{M}, \quad \tilde{p}(\tilde{x}) = p(x) \quad \text{and} \quad \tilde{p}(\tilde{b}) = p(b).$$

Furthermore, $\tilde{\Psi}(\tilde{y}, \hat{\tilde{Y}}) = \Psi(y, \hat{Y})$ by the definition in (D.1) and

$$\begin{aligned} \tilde{f}\left[\mathrm{e}^{-\delta}\tilde{\imath}(\tilde{x}, \tilde{y}) + \hat{\tilde{Y}}, \hat{\tilde{Y}}\right] &= \tilde{f}\left[\mathrm{e}^{-\delta}\sigma i(x, y) + \mu + \sigma\hat{Y}, \hat{\tilde{Y}}\right] \\ &= \tilde{f}\left[\mu + \sigma\left(\mathrm{e}^{-\delta} i(x, y) + \hat{Y}\right), \hat{\tilde{Y}}\right] = f\left[\mathrm{e}^{-\delta} i(x, y) + \hat{Y}, \hat{Y}\right]. \end{aligned}$$

The above analysis implies that (D.6) holds. To see that (D.7) holds, note that

$$\begin{aligned} x_{\tilde{f}}(\tilde{y}) &= \inf\left\{\mu + \sigma x \geq \mu - \frac{a\sigma}{d} : f(x, y) = 0\right\} \\ &= \mu + \sigma \inf\left\{x \geq p^{-1}(0) : f(x, y) = 0\right\} = \mu + \sigma x_f(y), \end{aligned}$$

where we have used the definition of p and $\tilde{p}$. This yields $\tilde{x} < x_{\tilde{f}}(\tilde{y})$ iff $x < x_f(y)$. In combination with (D.4), we obtain

$$\tilde{\imath}(\tilde{x}, \tilde{y}) = \sigma i(x, y) = \begin{cases} \sigma\left(x - p^{-1}[f(x, y)]\right), & \tilde{x} < x_{\tilde{f}}(\tilde{y}), \\ \sigma\left(x_f(y) - p^{-1}(0)\right), & \tilde{x} \geq x_{\tilde{f}}(\tilde{y}). \end{cases}$$

When $\tilde{x} < x_{\tilde{f}}(\tilde{y})$, using (D.5) and the definition of p and $\tilde{p}$, we obtain

$$\begin{aligned} \sigma\left(x - p^{-1}[f(x, y)]\right) &= \sigma x - \sigma p^{-1}[\tilde{f}(\tilde{x}, \tilde{y})] \\ &= \tilde{x} - \mu - \sigma\left(\frac{\tilde{f}(\tilde{x}, \tilde{y}) - a}{d}\right) = \tilde{x} - \tilde{p}^{-1}[\tilde{f}(\tilde{x}, \tilde{y})]. \end{aligned}$$

When $\tilde{x} \geq x_{\tilde{f}}(\tilde{y})$, using the definition of p and $\tilde{p}$ again yields

$$\sigma\left(x_f(y) - p^{-1}(0)\right) = x_{\tilde{f}}(\tilde{y}) - \mu + \frac{a\sigma}{d} = x_{\tilde{f}}(\tilde{y}) - \tilde{p}^{-1}(0).$$

The above analysis implies that (D.7) holds. Therefore, economies E and $\tilde{E}$ generate the same commodity price process. □

Appendix E. Algorithms

The storage model is solved by a modified version of the endogenous grid method of Carroll (2006). The candidate space $\mathscr{C}$ and $\bar{p}_f(z)$ are as defined in Appendix A. We derive the following property in order to handle free-disposal and state-dependent discounting in the numerical computation.

Lemma E.1. *For each f in the candidate space $\mathscr{C}$, if $x > p^{-1}[\bar{p}_f(z)]$, then*

$$Tf(x,z) = \max\left\{\min\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x,z)]\right) + \hat{Y}, \hat{Z}\right) - k, p(b)\right\}, 0\right\}.$$

If in addition $\bar{p}_f^0(z) \leq p(b)$, then

$$Tf(x,z) = \max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M}f\left(\mathrm{e}^{-\delta}\left(x - p^{-1}[Tf(x,z)]\right) + \hat{Y}, \hat{Z}\right) - k, 0\right\}.$$

Proof. The first statement is immediate by Lemma A.5 (ii)–(iii) and the fact that $Tf(x,z)$ is decreasing in x. The second statement follows from the definition of $\bar{p}_f^0(z)$ and the monotonicity of f in its first argument as a candidate in $\mathscr{C}$. □

E.1. The Endogenous Grid Algorithm.

We define a finite Markov Chain $\{Z_t\}$.[3] The states are indexed by j and m, and the transition matrix has elements $\Phi_{j,m}$. Moreover, we use $\mathcal{D}(x,z) = p^{-1}[f(x,z)]$ to denote a candidate equilibrium demand function. The endogenous grid algorithm for computing the equilibrium pricing rule is described in Algorithm 1.

In particular, we choose to approximate the demand function $\mathcal{D}(x,z)$ in Step 2 instead of the price function $f(x,z)$. This is helpful for improving both precision and stability of the algorithm when the demand function diverges at the lower bound of the endogenous state space. A typical example is the exponential demand $p(x) = x^{-1/\lambda} (\lambda > 0)$, which is commonly adopted by applied research (see, e.g., Deaton and Laroque, 1992; Gouel and Legrand, 2022). If the inverse demand function is linear as in Section 4, however, then it is innocuous to approximate the price function directly.

Moreover, the validity and convergence of the updating process in Step 3 are justified by Theorem A.1, Lemma A.5, and Lemma E.1 above.

[3]In the model with only the speculative channel, we discretize the interest rate process following Tauchen (1986) and use a Markov chain with $N = 101$ states. In the model with the demand channel, we discretize the VAR model representing the joint dynamics of interest rate and economic activity following Schmitt-Grohé and Uribe (2014).

Algorithm 1 The endogenous grid algorithm

Step 1. Initialization step. Choose a convergence criterion $\varpi > 0$, a grid on storage $\{I_s\}$ starting at 0, a grid on discount factor shocks for numerical integration $\{\varepsilon_l\}$ with associated weights ω_l, a grid on production shocks for numerical integration $\{\eta_n\}$ with associated weights w_n, and an initial policy rule (guessed): $\{X^1_{s,j}\}$ and $\{P^1_{s,j}\}$. Start iteration at $i = 1$.

Step 2. Update the demand function via linear interpolation and extrapolation:

$$p^{-1}\left(P^i_{s,j}\right) = \mathcal{D}^i\left(X^i_{s,j}, Z_j\right). \tag{E.1}$$

Step 3. Obtain prices and availability consistent with the grid of stocks and Markov Chain:

$$P^{i+1}_{s,j} = \max\left\{\min\left\{\tilde{P}^{i+1}_{s,j},\, p(b)\right\}, 0\right\} \quad \text{and} \quad X^{i+1}_{s,j} = I_s + p^{-1}\left(P^{i+1}_{s,j}\right), \tag{E.2}$$

where

$$\tilde{P}^{i+1}_{s,j} = \mathrm{e}^{-\delta} \sum_{l,m,n} \omega_l \Phi_{j,m} w_n m\left(Z_m, \varepsilon_l\right) p\left(\mathcal{D}^i\left(y\left(Z_m, \eta_n\right) + \mathrm{e}^{-\delta} I_s, Z_m\right)\right) - k.$$

Step 4. Terminal step. If $\max_{s,j} |P^{i+1}_{s,j} - P^i_{s,j}| \geq \varpi$ then increment i to $i+1$ and go to step 2. Otherwise, approximate the equilibrium pricing rule by $f^*(x,z) = p[\mathcal{D}^i(x,z)]$.

E.2. **Solution Precision.** To evaluate the precision of the numerical solution, we refer to a suitably adjusted version of the bounded rationality measure originally designed by Judd (1992), which we name as the Euler equation error and measures how much solutions violate the optimization conditions. In the current context, it is defined at state (x,z) as

$$EE_f(x,z) = 1 - \frac{\mathcal{D}_1(x,z)}{\mathcal{D}_2(x,z)},$$

where f is the numerical solution of the equilibrium price,

$$\mathcal{D}_1(x,z) = p^{-1}\left[\min\left\{\max\left\{\mathrm{e}^{-\delta}\,\mathbb{E}_z\,\hat{M} f(\hat{X},\hat{Z}) - k, p(x)\right\}, p(b)\right\}\right] - b$$

and $\mathcal{D}_2(x,z) = p^{-1}[f(x,z)] - b$. In particular, both $\mathcal{D}_1(x,z)$ and $\mathcal{D}_2(x,z)$ are expressed in terms of the *relative* demand for commodity, since b is the greatest lower bound (hence corresponds to the *zero* level) of the total available supply. Therefore, $EE_f(x,z)$ measures the error at state (x,z), in terms of the quantity consumed, incurred by using the numerical solution instead of the true equilibrium pricing rule.

To evaluate the precision of the endogenous grid algorithm in the context of Section 4.2.1, we simulate a time series $\{(X_t, R_t)\}_{t=1}^T$ of length T =20,000 based on the state evolution path $X_{t+1} = \mathrm{e}^{-\delta} i(X_t, R_t) + Y_t$ and $R_{t+1} \sim \Phi(R_t, \cdot)$, where (X_0, R_0) is given, and

$$i(X_t, R_t) = \min\{X_t, x^*_f(R_t)\} - p^{-1}[f(X_t, R_t)]$$

is the equilibrium storage function computed by the endogenous grid algorithm. We discard the first 1,000 draws, and then compute the Euler equation error at the truncated time series. When applying the endogenous grid algorithm, we use an exponential grid for storage in the range $[0,2]$ with median value 0.5, function iteration is implemented via linear interpolation and linear extrapolation, and we terminate the iteration process at precision $\varpi = 10^{-4}$. The rest of the setting is same to Section 4.

TABLE E.1. Precision under different grid sizes and different parameters

A. Different grid sizes									
	$K = 100$			$K = 200$			$K = 1{,}000$		
Precision	$N = 7$	$N = 51$	$N = 101$	$N = 7$	$N = 51$	$N = 101$	$N = 7$	$N = 51$	$N = 101$
max	−3.61	−3.64	−3.64	−3.96	−4.01	−4.02	−4.69	−5.21	−5.16
95%	−4.69	−4.66	−4.67	−5.44	−5.39	−5.39	−6.72	−6.76	−6.77
B. Different parameters									
	$\lambda = -0.03$			$\lambda = -0.06$			$\lambda = -0.15$		
Precision	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.05$	$\delta = 0.01$	$\delta = 0.02$	$\delta = 0.05$
max	−3.63	−3.65	−3.63	−3.64	−3.64	−3.59	−3.68	−3.68	−3.67
95%	−5.09	−4.9	−4.65	−4.9	−4.66	−4.45	−4.63	−4.52	−4.66

Notes: In Panel A, we fix $\lambda = -0.06$ and $\delta = 0.02$, simulate a time series of length $T = 20{,}000$, discard the first 1,000 draws, and then compute the level of precision as $\log_{10} |EE_f|$. When applying the endogenous grid algorithm, we use an exponential grid for storage in the range $[0, 2]$ with median value 0.5, function iteration is implemented via linear interpolation and linear extrapolation, and we terminate the iteration process at precision $\varpi = 10^{-4}$. The rest of the setting is same to Section 4.2.1. In Panel B, we fix the grid size to $K = 100$ and $N = 51$, and vary the parameters.

Summary statistics (maximum as well as 95-th percentile) are reported in Table E.1, where K is the number of grid points for storage, N is the number of state points for interest rates, and precision at (x, z) is evaluated as $\log_{10} |EE_f(x, z)|$. The results demonstrate that the endogenous grid algorithm attains a high level of precision, with an Euler equation error uniformly less than 0.025%.

E.3. **The Generalized Impulse Response Function.** To properly capture the nonlinear asymmetric dynamics of the competitive storage model and effectively study the dynamic causal effect of interest rates on commodity prices, we refer to the generalized impulse response function proposed by Koop et al. (1996), which defines IRFs as state-and-history-dependent random variables and is applicable to both linear and nonlinear multivariate models. We are interested in calculating the IRFs when (X_{t-1}, Z_{t-1}) are held at different percentiles of the stationary distribution.

Algorithm 2 clarifies the computation process of the generalized IRFs based on the setting of Section 4. However, the algorithm can be easily extended to handle more general settings as formulated in Section 2, where more advanced interest rate and production setups are allowed. To proceed, we define

$$F(x, z, Y) := \mathrm{e}^{-\delta} \left(\min\{x, x^*(z)\} - p^{-1}[f^*(x, z)] \right) + Y.$$

Algorithm 2 The generalized impulse response function

Step 1. Initialization step. Choose initial values for X_{t-1} and Z_{t-1}, and a finite horizon H and a size of Monte Carlo samples N. Furthermore, set the initial samples as

$$\tilde{X}^n_{t-1} = X^n_{t-1} \equiv X_{t-1} \quad \text{and} \quad Z^n_{t-1} = \tilde{Z}^n_{t-1} \equiv Z_{t-1}.$$

Step 2. Randomly sample $(H+1) \times N$ values of production shocks $\left\{Y^{S,n}_{t+h}\right\}^{(H,N)}_{(h,n)=(0,1)}$.

Step 3. (Baseline Economy) Sample $(H+1) \times N$ values of the exogenous states and calculate the net production

$$\{Z^n_{t+h}\}^{(H,N)}_{(h,n)=(0,1)} \quad \text{where} \quad Z^n_{t+h} \sim \Pi(Z^n_{t+h-1}, \cdot\,),$$

$$\{Y^n_{t+h}\}^{(H,N)}_{(h,n)=(0,1)} \quad \text{where} \quad Y^n_{t+h} = y\left(Z^n_{t+h}, Y^{S,n}_{t+h}\right).$$

Step 4. (Impulse Shock Economy) Compute the period-t exogenous states $\{\tilde{Z}^n_t\}^N_{n=1}$ after the shock. Sample $H \times N$ values of the exogenous states and calculate the net production

$$\{\tilde{Z}^n_{t+h}\}^{(H,N)}_{(h,n)=(1,1)} \quad \text{where} \quad \tilde{Z}^n_{t+h} \sim \Pi(\tilde{Z}^n_{t+h-1}, \cdot\,),$$

$$\{\tilde{Y}^n_{t+h}\}^{(H,N)}_{(h,n)=(0,1)} \quad \text{where} \quad \tilde{Y}^n_{t+h} = y\left(\tilde{Z}^n_{t+h}, Y^{S,n}_{t+h}\right).$$

Step 5. For $h = 0, \dots, H$ and $n = 1, \dots, N$, compute the sequence of availability

$$X^n_{t+h} = F(X^n_{t+h-1}, Z^n_{t+h-1}, Y^n_{t+h}) \quad \text{and} \quad \tilde{X}^n_{t+h} = F(\tilde{X}^n_{t+h-1}, \tilde{Z}^n_{t+h-1}, \tilde{Y}^n_{t+h}).$$

Step 6. For $h = 0, \dots, H$, compute the period-$(t+h)$ impulse response

$$IRF(t+h) = \frac{1}{N}\sum_{n=1}^{N} f^*(\tilde{X}^n_{t+h}, \tilde{Z}^n_{t+h}) - \frac{1}{N}\sum_{n=1}^{N} f^*(X^n_{t+h}, Z^n_{t+h}).$$

The stationary distribution of the state process is computed based on ergodicity. Once f^*, i^*, and x^* are calculated, we simulate a time series of $\{(X_t, Z_t)\}^T_{t=1}$ according to

$$X_{t+1} = \mathrm{e}^{-\delta}\left(\min\{X_t, x^*(Z_t)\} - p^{-1}[f^*(X_t, Z_t)]\right) + y\left(Z_{t+1}, Y^S_{t+1}\right) \text{ and } Z_{t+1} \sim \Pi(Z_t, \cdot)$$

for $T = 200{,}000$ periods, discard the first 50,000 samples, and use the remainder to approximate the stationary distribution.

Appendix F. MIT shock in a Constant Interest Rate Model

Consider the classical competitive storage model of Deaton and Laroque (1992). As in their paper, we assume that gross interest rates are constant at $r_\ell > 1$, and output is IID and satisfies $Y_t = y(\eta_t)$. Denote this as economy 1.

Consider an alternative economy, in which an MIT shock increases interest rates from r_ℓ to r_h at the beginning of period t, and interest rates return to r_ℓ in all subsequent periods. Denote this as economy 2.

Let f^*_j be the equilibrium pricing rule, i^*_j be the equilibrium storage, x^*_j be the equilibrium free-disposal threshold, and T_j be the equilibrium price operator when interest rates are constant at r_j for $j \in \{\ell, h\}$.

Lemma F.1. *$T_\ell f \geq T_h f$ for all $f \in \mathscr{C}$. Moreover, $f^*_\ell \geq f^*_h$ and $x^*_\ell \geq x^*_h$.*

Proof. Since interest rates are constant and the output process is IID, the equilibrium objects are functions of only the endogenous state. Fix $f \in \mathscr{C}$. Denote x^*_{fj} as the free-disposal threshold (related to candidate f) when interest rates are constant at r_j for $j \in \{\ell, h\}$. Since $r_\ell < r_h$, we have

$$\frac{\mathrm{e}^{-\delta}}{r_\ell} \mathbb{E} f\left(\mathrm{e}^{-\delta}[x^*_{f\ell} - p^{-1}(0)] + \hat{Y}\right) \geq \frac{\mathrm{e}^{-\delta}}{r_h} \mathbb{E} f\left(\mathrm{e}^{-\delta}[x^*_{f\ell} - p^{-1}(0)] + \hat{Y}\right).$$

By the continuity and monotonicity of f and the definition of the free-disposal threshold, we have $x^*_{f\ell} \geq x^*_{fh}$. In particular, since f is chosen arbitrarily and $f^* \in \mathscr{C}$, we have $x^*_\ell \geq x^*_h$.

Next, we show that $T_\ell f \geq T_h f$. Since $T_h f(x) = 0$ whenever $x \geq x^*_{fh}$ by Lemma A.5 and $T_\ell f \geq 0$, it suffices to verify that $T_\ell f(x) \geq T_h f(x)$ for all $x < x^*_{fh}$. Suppose on the contrary that $\xi_1 := T_\ell f(x) < T_h f(x) =: \xi_2$ for some $x < x^*_{fh}$. By the definition of the equilibrium price operator, we have

$$\begin{aligned}
\xi_1 &= \min\left\{\max\left\{\frac{\mathrm{e}^{-\delta}}{r_\ell} \mathbb{E} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi_1)] + \hat{Y}\right) - k, p(x)\right\}, p(b)\right\} \\
&\geq \min\left\{\max\left\{\frac{\mathrm{e}^{-\delta}}{r_\ell} \mathbb{E} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi_2)] + \hat{Y}\right) - k, p(x)\right\}, p(b)\right\} \\
&\geq \min\left\{\max\left\{\frac{\mathrm{e}^{-\delta}}{r_h} \mathbb{E} f\left(\mathrm{e}^{-\delta}[x - p^{-1}(\xi_2)] + \hat{Y}\right) - k, p(x)\right\}, p(b)\right\} = \xi_2,
\end{aligned}$$

which is a contradiction. Hence we have shown that $T_\ell f \geq T_h f$ for all $f \in \mathscr{C}$.

Suppose $T^{n-1}_\ell f \geq T^{n-1}_h f$. Since the equilibrium price operator is order-preserving by Lemma A.2, we have $T^n_\ell f = T_\ell(T^{n-1}_\ell f) \geq T_\ell(T^{n-1}_h f) \geq T^n_h f$. By induction, we have shown that $T^n_\ell f \geq T^n_h f$ for all n. Because $T^n_\ell f \to f^*_\ell$ and $T^n_h f \to f^*_h$ by Theorem A.1, letting $n \to \infty$ gives $f^*_\ell \geq f^*_h$. □

The next result indicates that, in the classical competitive storage model, an MIT interest rate shock only has a contemporaneous negative effect on commodity prices, which will die out immediately starting from the next period. In each future period, commodity prices are at least as large as their level when shocks are absent.

Proposition F.1 (MIT Shock). *If $X^1_{t-1} = X^2_{t-1}$ with probability one, then $P^1_t \geq P^2_t$ and $P^1_{t+h} \leq P^2_{t+h}$ with probability one for all $h \geq 1$.*

Proof. Since shocks are unexpected and both economies share the same interest rate r_ℓ in period $t-1$ and the same output $Y_t = y(\eta_t)$ in each period, we have

$$X^1_t = \mathrm{e}^{-\delta} i^*_\ell(X^1_{t-1}) + Y_t = \mathrm{e}^{-\delta} i^*_\ell(X^2_{t-1}) + Y_t = X^2_t$$

with probability one. Since in period t interest rate in economy 2 is higher, agents will account for the temporarily different incentive, hence the price function in economy 2 is $f^*_{h\ell} = T_h f^*_\ell$ and the inventory function is

$$i^*_{h\ell}(x) = \min\{x^*_\ell, x\} - p^{-1}[f^*_{h\ell}(x)]$$

Based on Lemma F.1, we have

$$f_{h\ell}^* = T_h f_\ell^* \leq T_\ell f_\ell^* = f_\ell^*$$

and thus

$$i_{h\ell}^*(x) \leq \min\{x_\ell^*, x\} - p^{-1}[f_\ell^*(x)] = i_\ell^*(x).$$

As we have shown that $X_t^1 = X_t^2$ with probability one, the above inequalities imply that

$$P_t^1 = f_\ell^*(X_t^1) = f_\ell^*(X_t^2) \geq f_{h\ell}^*(X_t^2) = P_t^2$$

and

$$X_{t+1}^1 = \mathrm{e}^{-\delta} i_\ell^*(X_t^1) + Y_{t+1} \geq \mathrm{e}^{-\delta} i_{h\ell}^*(X_t^2) + Y_{t+1} = X_{t+1}^2$$

with probability one. Since starting from period $t+1$ interest rates return to r_ℓ, the equilibrium objects become f_ℓ^* and i_ℓ^* in both economies. By the monotonicity of f_ℓ^* and i_ℓ^*, we have

$$P_{t+1}^1 = f_\ell^*(X_{t+1}^1) \leq f_\ell^*(X_{t+1}^2) = P_{t+1}^2$$

and

$$X_{t+2}^1 = \mathrm{e}^{-\delta} i_\ell^*(X_{t+1}^1) + Y_{t+2} \geq \mathrm{e}^{-\delta} i_\ell^*(X_{t+1}^2) + Y_{t+2} = X_{t+2}^2$$

with probability one. By induction, we can then show that $P_{t+h}^1 \leq P_{t+h}^2$ with probability one for all $h \geq 1$. □

Appendix G. A Necessity Result for Discounting

In this section, we show that Assumption 2.1 is necessary in a range of standard settings: no equilibrium solution exists if Assumption 2.1 fails. Throughout, we focus on the case where

$$b = 0, \quad \mathsf{X} = (b, \infty) \quad \text{and} \quad M_t = m(Z_t).$$

Let $\mathscr{B}_\mathsf{S}$ be the Borel subsets of S. Let $\mathcal{K}(\mathsf{S}, \mathsf{X})$ be the set of all stochastic kernels $\Psi(x, z, \mathrm{d}\, x')$ from S to X such that

$$Qf(x,z) := \sum_{z'} \int f(x', z') \Phi(z, z') \Psi(x, z, \mathrm{d}\, x'), \qquad (x,z) \in \mathsf{S}$$

has a stationary distribution π on the set of Borel probability measures on S and is irreducible and weakly compact as an operator on $L_1 := L_1(\mathsf{S}, \mathscr{B}_\mathsf{S}, \pi)$.

Given $\Psi \in \mathcal{K}(\mathsf{S}, \mathsf{X})$, we consider the functional equation

$$f(x,z) = \max\left\{ \mathrm{e}^{-\delta} \sum_{z'} m(z') \int f(x', z') \Phi(z, z') \Psi(x, z, \mathrm{d}\, x'), p(x) \right\}, \tag{G.1}$$

where m is a positive function on Z and p is a decreasing map from X to itself with $\int p \,\mathrm{d}\, \pi < \infty$ and $p(x) \uparrow \infty$ as $x \downarrow 0$. Letting K be the positive linear operator on L_1 defined by

$$Kf(x,z) := \mathrm{e}^{-\delta} \sum_{z'} m(z') \int f(x', z') \Phi(z, z') \Psi(x, z, \mathrm{d}\, x'), \qquad (x,z) \in \mathsf{S},$$

we can also write (G.1) as $f = Kf \vee p$. Same as above, let $s(K)$ be the spectral radius of K as a linear operator on L_1.

Lemma G.1. *If* $\Psi \in \mathcal{K}(\mathsf{S}, \mathsf{X})$*, then* $-\ln s(K) = \delta + \kappa(M)$*.*

Proof. An induction argument shows that

$$K^n \mathbb{1}(x, z) = \mathrm{e}^{-\delta n} \, \mathbb{E}_z \prod_{t=1}^{n} M_t.$$

Letting (X_0, Z_0) be a draw from π, we obtain

$$\|K^n \mathbb{1}\| = \mathbb{E} \, K^n \mathbb{1}(X_0, Z_0) = \mathrm{e}^{-\delta n} \, \mathbb{E} \left[\mathbb{E}_{Z_0} \prod_{t=1}^{n} M_t \right] = \mathrm{e}^{-\delta n} \, \mathbb{E} \prod_{t=1}^{n} M_t.$$

Hence, by weak compactness of K (which implies compactness of K^2 by Theorem 9.9 of Schaefer, 1974) and Theorem B2 of Borovička and Stachurski (2020), we have

$$s(K) = \lim_{n \to \infty} \|K^n \mathbb{1}\|^{1/n} = \lim_{n \to \infty} \left\{ \mathrm{e}^{-\delta n} \, \mathbb{E} \prod_{t=1}^{n} M_t \right\}^{1/n} = \mathrm{e}^{-\delta} \lim_{n \to \infty} q_n^{1/n}.$$

It follows that $-\ln s(K) = \delta + \kappa(M)$, as was to be shown. □

The following result demonstrates the necessity of Assumption 2.1 in the above standard setting.

Proposition G.1. *If there exists an* $f \in L_1$ *and* $\Psi \in \mathcal{K}(\mathsf{S}, \mathsf{X})$ *such that* (G.1) *holds, then* $\delta + \kappa(M) \geq 0$*. If, in addition,*

$$E := \{(x, z) \in \mathsf{S} : Kf(x, z) < p(x)\}$$

obeys $\pi(E) > 0$*, then* $\delta + \kappa(M) > 0$*.*

Proof. Let f and h have the stated properties. Regarding the first claim, we note that, since Q is weakly compact and irreducible on L_1, and since m is positive and bounded, the operator K is likewise weakly compact and irreducible. By the Krein–Rutman theorem, combined with irreducibility and weak compactness of K (see, in particular, Lemma 4.2.11 of Meyer-Nieberg, 2012), there exists an $e \in L_\infty := L_\infty(\mathsf{S}, \mathscr{B}_{\mathsf{S}}, \pi)$ such that $e > 0$ π-almost everywhere and $K^* e = s(K) e$, where K^* is the adjoint of K.

By (G.1) we have $f = Kf \vee p$, so $f \geq Kf$. Hence

$$\langle e, f \rangle \geq \langle e, Kf \rangle = \langle K^* e, f \rangle = s(K) \langle e, f \rangle. \tag{G.2}$$

Since $f \geq p > 0$ and e is positive π-a.e., we have $\langle e, f \rangle > 0$. Hence $s(K) \leq 1$. Applying Lemma G.1 now yields $\delta + \kappa(M) \geq 0$.

Regarding the second claim, suppose that $\pi(E) > 0$, where E is as defined in Proposition G.1. It then follows from $f = Kf \vee p$ that $f > Kf$ on a set of positive π-measure. But then, since e is positive π-a.e., we have $\langle e, f \rangle > \langle e, Kf \rangle = s(K) \langle e, f \rangle$, where the equality is from (G.2). As before we have $\langle e, f \rangle > 0$, so $s(K) < 1$. Using Lemma G.1 again we obtain $\delta + \kappa(M) > 0$. □

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